# MATH 3235 Probability Theory

10/04/2022

c.d.f. (d.f.)

$X$ is a r.v.

$$F(x) = \mathbb{P}(X \leq x)$$

$$\mathbb{P}(b < X \leq a) = F(a) - F(b)$$

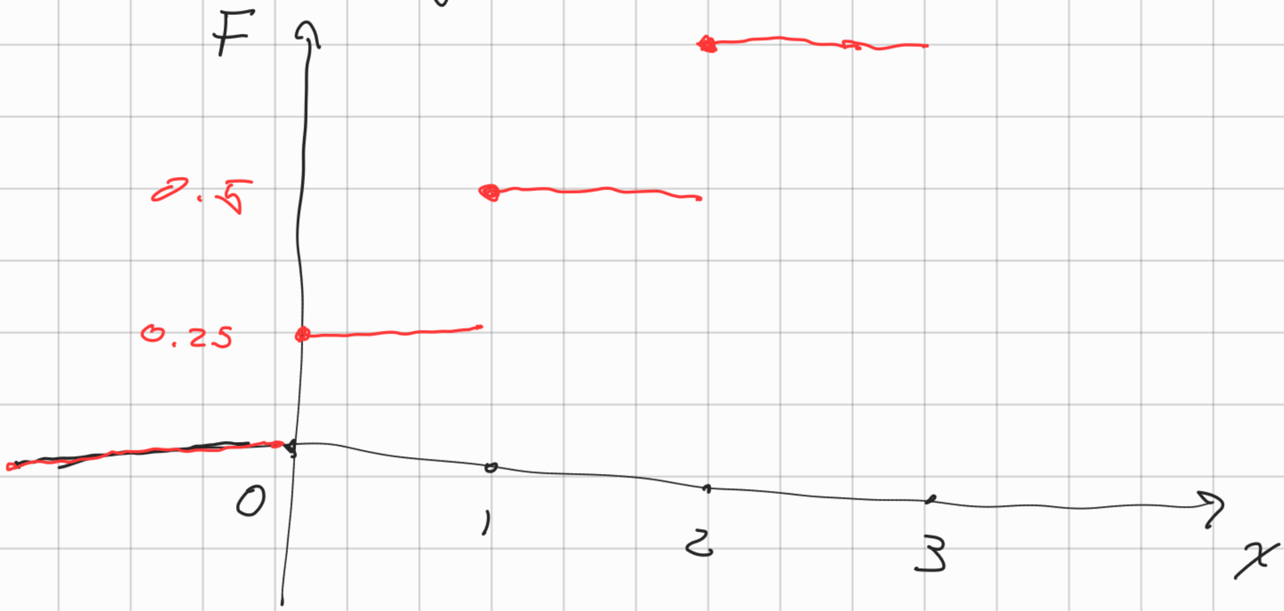
$$\mathbb{P}(b \leq X \leq a) =$$

$$[b, a] = \bigcap_n \left(b - \frac{1}{n}, a\right]$$

$$\mathbb{P}(b \leq X \leq a) = \lim_{n \to \infty} \mathbb{P}\left(b - \frac{1}{n} < X \leq a\right)$$

$$= F(a) - \lim_{n \to \infty} F\left(b - \frac{1}{n}\right).$$

What if F is discrete




$$\lim_{x \to -\infty} F(x) = 0$$

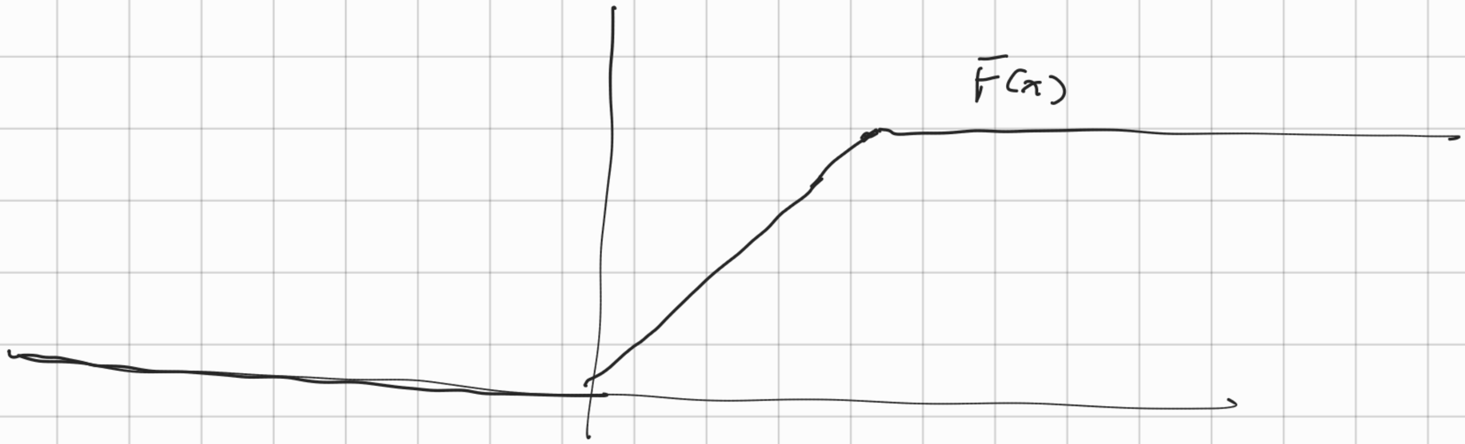
$$\lim_{x \to +\infty} F(x) = 1$$

$F(x)$ non decreasing

$F(x)$ is continuos from the right

$$\lim_{\varepsilon \to 0^+} F(x+\varepsilon) = F(x)$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$




if $0 \le a,\ b \le 1 \implies$

$$\mathbb{P}(a \le X \le b) = b - a$$

Uniform r.v. in $[0,1]$

$$\mathbb{P}(X = x) = \lim_{\varepsilon \to 0} \mathbb{P}\left(x - \varepsilon < X \le x\right) =$$

$$= F(x) - \lim_{\varepsilon \to 0^+} F(x - \varepsilon) = 0$$

$$\frac{\mathbb{P}(x \le X \le x + dx)}{dx} = \frac{dx}{dx} = 1$$

if $0 \le x \le 1$.

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$f(x)$ p.d.f.

$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx$$

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→ If $F(x)$ is continoous then

$\mathbb{P}(X = x) = 0 \quad \forall x$.

Non atomic.

→ if $F'(x)$ exists but for a

finite number of point

$X$ is continuous and

$$f(x) = F'(x)$$

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy$$

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$[A, B]$

$$f(x) = \begin{cases} 0 & x < A \\ C & A \leq x \leq B \\ 0 & x > B \end{cases}$$

$$\underbrace{F(+\infty)}_{1} - \underbrace{F(-\infty)}_{0} = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

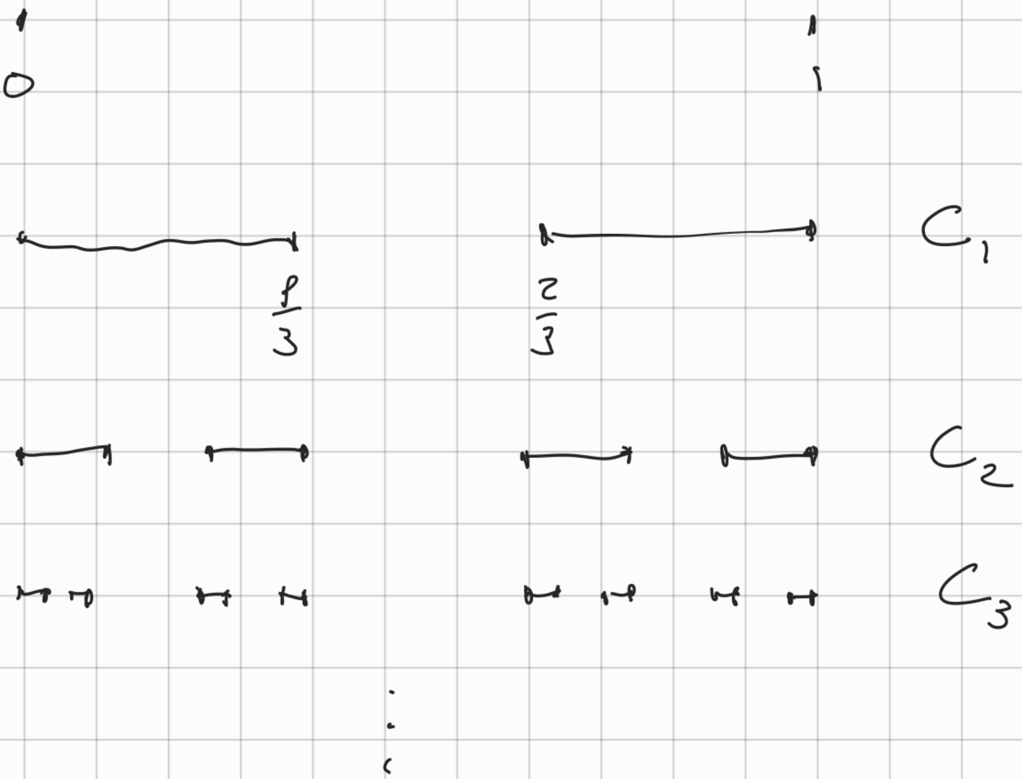
$$= \int_A^B C\,dx = (B - A)C = 1$$

$$C = \frac{1}{B-A}$$

$$f(x) = \begin{cases} 0 & x < A \\ \frac{1}{B-A} & A \le x \le B \\ 0 & x > B \end{cases}$$

$$F(x) = \begin{cases} 0 & x \le A \\ \frac{x}{B-A} & A \le x \le B \\ 1 & x \ge B \end{cases}$$

Cantor set.



$$C = \bigcap_{i=1}^{\infty} C_i$$

$x \in [0, 1]$ $\qquad x = \sum_{i=1}^{\infty} a_i \, 3^{-i}$

with $a_i \in \{0, 1, 2\}$

$C$ is the set of all points for which $a_i \in \{0, 2\}$

$C$ is not countable

$$F(x) = \sum_{i=1}^{\infty} \frac{a_i/2}{2^i}$$

if $x \in C \Rightarrow a_i/2 \in \{0, 1\}$

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$\infty$

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Total length of

$C_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$C_2 = 4/9$

$\vdots$

$$C_n = \left(\frac{2}{3}\right)^n$$

$C$ has measure $0$.

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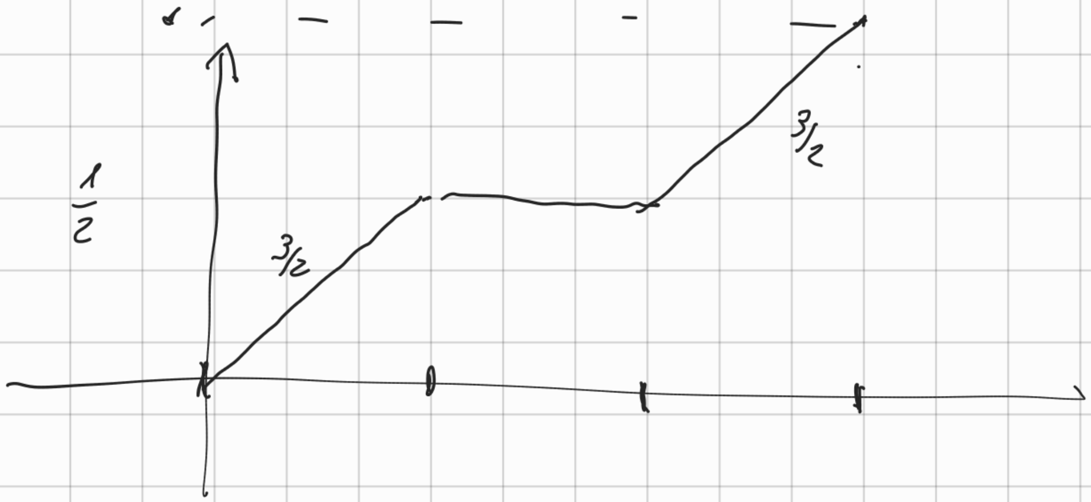
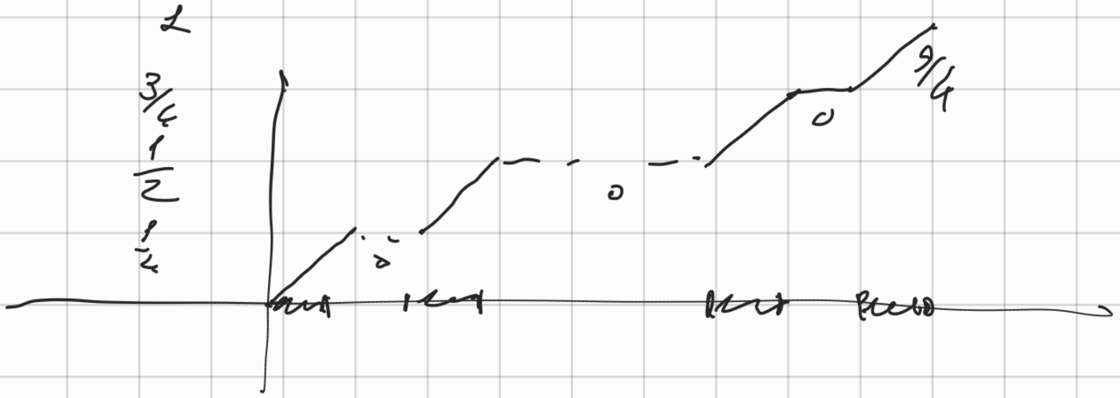
$C_1$

$F_1(x)$




For every $n$ I get a c.d.f. $F_n(x)$

$\lim_{n \to \infty} F_n(x) = F(x)$ and is continuous

$$F(x) = \sum \frac{a_i/2}{2^i} \quad \text{for} \quad x \in C$$

Devil stair case.

Take $x$ at random $[0,1]$

$x \notin C$ with prob 1

$F'(x)$ exists and it is $0$.

$F(0) = 0 \qquad F(1) = 1.$

$$\int_0^1 F'(x)\,dx \neq F(1) - F(0)$$

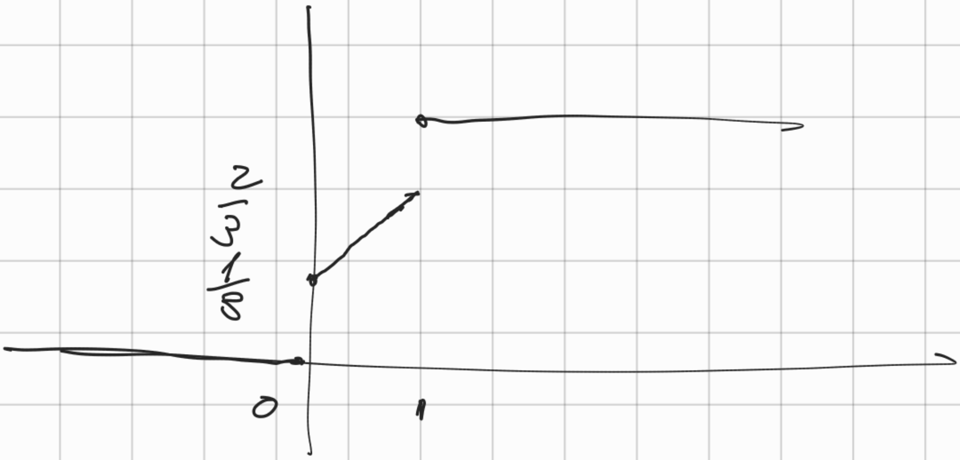
$X$ is a r.v. with c.d.f.

The devil staircase $\Rightarrow$

$X$ is not atomic but it has

no density.

F




c.d.f.

not continuous, not discrete

$$\mathbb{P}(X=0)=\frac{1}{3}$$

$$\mathbb{P}(X=1)=\frac{1}{3}$$

$$\mathbb{P}(a\le X\le b)\le \frac{b-a}{3}$$

$$F(x)=F_c(x)+F_d(x).$$

$f(x)$ is a density function with a finite number of discontinuity.

$F(x)$ is continuous everywhere and differentiable but for finitely many points

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$$\mathbb{P}(x \le X \le x + dx) = f(x)\,dx$$

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Some Example.

Uniform in $[A, B]$

Exponential r.v.

$$f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy =$$

$$= \lambda \int_0^x e^{-\lambda y}\,dy = e^{-\lambda y}\Big|_0^x =$$

$$= 1 - e^{-\lambda x}$$

The first H flipping a coin

$q^k p$ of flippind k Times

flip N Times in a second

$p \to \frac{\lambda}{N}$

How many second Till 1st H

$\left(1 - \frac{\lambda}{N}\right)^{tN} \frac{\lambda}{N}$

t seconds

$\mathbb{P}$( first arrival is between t and

$t + \frac{1}{N}) = \left(1 - \frac{\lambda}{N}\right)^{tN} \frac{\lambda}{N}$

$f(t) = \lambda e^{-\lambda t}$